Eur. Phys. J. B **38**, 635–649 (2004)
DOI: 10.1140/epjb/e2004-00158-1

THE EUROPEAN PHYSICAL JOURNAL B

Real Hamiltonian forms of Hamiltonian systems

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Received 8 October 2003
Published online 8 June 2004 –

Abstract. We introduce the notion of a real form of a Hamiltonian dynamical system in analogy with the notion of real forms for simple Lie algebras. This is done by restricting the complexified initial dynamical system to the fixed point set of a given involution. The resulting subspace is isomorphic (but not symplectomorphic) to the initial phase space. Thus to each real Hamiltonian system we are able to associate another nonequivalent (real) ones. A crucial role in this construction is played by the assumed analyticity and the invariance of the Hamiltonian under the involution. We show that if the initial system is Liouville integrable, then its complexification and its real forms will be integrable again and this provides a method of finding new integrable systems starting from known ones. We demonstrate our construction by finding real forms of dynamics for the Toda chain and a family of Calogero–Moser models. For these models we also show that the involution of the complexified phase space induces a Cartan-like involution of their Lax representations.

PACS. 02.30.Ik Integrable systems – 45.20.Jj Lagrangian and Hamiltonian mechanics

1 Introduction

Recently the so-called complex Toda chain (CTC) was shown to describe N-soliton interactions in the adiabatic approximation [1–3]. The complete integrability of the CTC is a direct consequence of the integrability of the real (standard) Toda chain (TC); it was also shown that CTC allows several dynamical regimes that are qualitatively different from the one of real Toda chain [2]. These results as well as the hope to understand the algebraic structures lying behind the integrability of CTC (such as, e.g. Lax representation) were the stimulation for the present work.

We start from a standard (real) Hamiltonian system $\mathcal{H} \equiv \{\omega, H, \mathcal{M}\}$ with n degrees of freedom and Hamiltonian H depending analytically on the dynamical variables. It is known that such systems can be complexified and then written as a Hamiltonian system with $2n$ (real) degrees of freedom. Our main aim is to show that to each compatible involutive automorphism $\widetilde{\mathcal{C}}$ of the complexified phase space we can relate a real Hamiltonian form of the initial system with n degrees of freedom. Just like to each complex Lie algebra one associates several inequivalent real forms, so to each $\mathcal{H}$ we associate several inequivalent real forms $\mathcal{H}_{\mathbb{R}} \equiv \{\omega_{\mathbb{R}}, H_{\mathbb{R}}, \mathcal{M}_{\mathbb{R}}\}$. Like the initial system $\mathcal{H}$, the real form is defined on a manifold $\mathcal{M}_{\mathbb{R}}$ with n real degrees of freedom. Provided $\widetilde{\mathcal{C}}(H) = H$ the dynamics on the real form will be well defined and will coincide with the dynamics on $\mathcal{M}_{\mathbb{C}}$ restricted to $\mathcal{M}_{\mathbb{R}}$. We show that if the initial system $\mathcal{H}$ is integrable then its real Hamiltonian forms will also be integrable. We pay special attention to the connection with integrable systems and the possibility they offer to define a class of new integrable systems starting from an initial one. Recently a procedure to obtain new integrable systems by composing known integrable ones has been elaborated in the framework of coproducts [4]. Here we are not concerned with this approach.

Examples of indefinite-metric Toda chain (IMTC) have already been studied by Kodama and Ye [5]. In particular they note that while the solutions of the TC model are regular for all t, the solutions of the IMTC model develop singularities for finite values of t. Particular examples of non-standard (or "twisted") real forms of $1+1$-dimensional Toda field theories have already been studied by Evans and Madsen [6] in connection with the problem of positive kinetic energy terms in the Lagrangian description and with emphasis on conformal WZNW models.

The approach we follow here is different and more general than the ones in [5,6]. Its main ideas were reported in [7]; here we elaborate the proofs the details and provide new classes of examples.

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2 Complexified Hamiltonian dynamics

We start with a real Hamiltonian system with n degrees of freedom $\mathcal{H} \equiv \left\{\mathcal{M}^{(n)}, H, \omega\right\}$ where $\mathcal{M}^{(n)}$ is a $2n$ dimensional vector space and

$$\omega = \sum_{k=1}^{n} dp_k \wedge dq_k. \tag{1}$$

Let's consider its complexification:

$$\mathcal{H}^{\mathbb{C}} \equiv \left\{\mathcal{M}^{\mathbb{C}}, H^{\mathbb{C}}, \omega^{\mathbb{C}}\right\} \tag{2}$$

where $\mathcal{M}_{\mathbb{C}}^{(2n)}$ can be viewed as a linear space $\mathcal{M}^{(n)}$ over the field of complex numbers:

$$\mathcal{M}_{\mathbb{C}}^{(2n)} = \mathcal{M}^{(n)} \oplus i\mathcal{M}^{(n)}.$$

In other words the dynamical variables p_k, q_k in $\mathcal{M}_{\mathbb{C}}^{(n)}$ now may take complex values. We assume that observables F, G and the Hamiltonian H are real analytic functions on $\mathcal{M}$ and can naturally be extended to $\mathcal{M}_{\mathbb{C}}^{(2n)}$.

The complexification of the dynamical variables F, G and H means that they become analytic functions of the complex arguments:

$$p_k{}^{\mathbb{C}} = p_{k,0} + ip_{k,1}, \;\; q_k{}^{\mathbb{C}} = q_{k,0} + iq_{k,1}, \;\; k = 1, \ldots, n \tag{3}$$

and we can write:

$$H^{\mathbb{C}} = H(p_k^{\mathbb{C}}, q_k^{\mathbb{C}}) = H_0 + iH_1. \tag{4}$$

The same goes true also for the complexified 2–form:

$$\omega^{\mathbb{C}} = \sum_{k=1}^{n} dp_k^{\mathbb{C}} \wedge dq_k^{\mathbb{C}} = \omega_0 + i\omega_1. \tag{5}$$

Note that each of the symplectic forms ω_0 and ω_1 are non-degenerate. However the linear combination $\omega^{\mathbb{C}} = \omega_0 + i\omega_1$ can be written down in the form:

$$\omega = \sum_{k=1}^{n} (dp_{k,0}, dp_{k,1}, dq_{k,0}, dq_{k,1})\, B_0 \begin{pmatrix} dp_{k,0} \\ dp_{k,1} \\ dq_{k,0} \\ dq_{k,1} \end{pmatrix}$$

where the matrix B_0

$$B_0 = \frac{1}{2} \begin{pmatrix} 0 & 0 & 1 & i \\ 0 & 0 & i & -1 \\ -1 & -i & 0 & 0 \\ -i & 1 & 0 & 0 \end{pmatrix}$$

obviously has the property $B_0^2 = 0$.

Remark 1. The kernel of $\omega^{\mathbb{C}}$ is spanned by the antiholomorphic vector fields. We could also choose the antiholomorphic (anti-analytic) functions in the complexification procedure. This would lead to equivalent results.

Obviously, $\dim \mathcal{M}^{\mathbb{C}} = 4n$ and therefore $\mathcal{H}^{\mathbb{C}}$ may be considered as a real dynamical system with $2n$ degrees of freedom. To elaborate on this, we start from the complexified equations of motion:

$$\frac{dp_k^{\mathbb{C}}}{dt} = -\frac{\partial H^{\mathbb{C}}}{\partial q_k^{\mathbb{C}}}, \qquad \frac{dq_k^{\mathbb{C}}}{dt} = \frac{\partial H^{\mathbb{C}}}{\partial p_k^{\mathbb{C}}}. \tag{6}$$

The right hand side of (6) contain the partial derivatives of both H_0 and H_1. Since we assumed analyticity, H_0 and H_1 will satisfy the Cauchy-Riemann equations:

$$\frac{\partial H_0}{\partial q_{k,0}} = \frac{\partial H_1}{\partial q_{k,1}}, \qquad \frac{\partial H_0}{\partial q_{k,1}} = -\frac{\partial H_1}{\partial q_{k,0}} \tag{7}$$

which means that the derivatives in the right hand sides of (6) are equal to:

$$\frac{\partial H^{\mathbb{C}}}{\partial q_k^{\mathbb{C}}} = \frac{\partial H_0}{\partial q_{k,0}} - i\frac{\partial H_0}{\partial q_{k,1}} = \frac{\partial H_1}{\partial q_{k,1}} + i\frac{\partial H_1}{\partial q_{k,0}}. \tag{8}$$

Analogous formulae hold for the derivatives with respect to $p_{k,0}$ and $p_{k,1}$. Thus all terms in the right hand sides of (6) can be expressed through the partial derivatives of H_0 only:

$$\frac{dp_{k,0}}{dt} = -\frac{\partial H_0}{\partial q_{k,0}}, \qquad \frac{dq_{k,0}}{dt} = \frac{\partial H_0}{\partial p_{k,0}}, \qquad \frac{dp_{k,1}}{dt} = \frac{\partial H_0}{\partial q_{k,1}}, \qquad \frac{dq_{k,1}}{dt} = -\frac{\partial H_0}{\partial p_{k,1}}. \tag{9}$$

Obviously (9) are standard Hamiltonian equations of motion for a dynamical system with $2n$ degrees of freedom corresponding to:

$$H_0 = \operatorname{Re} H^{\mathbb{C}}(p_k^{\mathbb{C}}, q_k^{\mathbb{C}}), \tag{10}$$
$$\omega_0 = \operatorname{Re} \omega^{\mathbb{C}} = \sum_{k=1}^{n} \left(dp_{k,0} \wedge dq_{k,0} - dp_{k,1} \wedge dq_{k,1}\right).$$

We denote the related real dynamical vector field by Γ_0.

The system (6) allows a second Hamiltonian formulation with:

$$H_1 = \operatorname{Im} H^{\mathbb{C}}(p_k^{\mathbb{C}}, q_k^{\mathbb{C}}), \tag{11}$$
$$\omega_1 = \operatorname{Im} \omega^{\mathbb{C}} = \sum_{k=1}^{n} \left(dp_{k,0} \wedge dq_{k,1} + dp_{k,1} \wedge dq_{k,0}\right)$$

and also real dynamical vector field Γ_1. Due to the analyticity of $H^{\mathbb{C}}$ Cauchy-Riemann equations yield that these two vector fields actually coincide:

$$\Gamma_0 = \Gamma_1\,.$$

So Γ_0 is a bi-Hamiltonian vector field and the corresponding recursion operator is:

$$T = \omega_0^{-1} \circ \omega_1 \tag{12}$$
$$= -\frac{\partial}{\partial p_1} \otimes dp_0 + \frac{\partial}{\partial p_0} \otimes dp_1 - \frac{\partial}{\partial q_1} \otimes dq_0 + \frac{\partial}{\partial q_0} \otimes dq_1$$
$$T^2 = -1.$$

3 Complexification and Liouville integrability

Let us now assume that our initial system is Liouville integrable and analyze what consequences will have this on the complexified system.

Proposition 1. *Let the initial system $\mathcal{H}$ have n functionally independent integrals I_k, $k = 1, \dots, n$ which are real analytic (meromorphic) functions of q_k and p_k. Let also I_k be in involution:*

$$\{I_k, I_j\} \equiv \sum_{s=1}^{n} \left(\frac{\partial I_k}{\partial p_s} \frac{\partial I_j}{\partial q_s} - \frac{\partial I_k}{\partial q_s} \frac{\partial I_j}{\partial p_s} \right) = 0, \qquad (13)$$

Then the complexified system $\left\{\mathcal{M}^{\mathbb{C}}, H_0, \omega_0\right\}$ is also Liouville integrable, i.e. it has $2n$ functionally independent integrals $I_{k,0}$, $I_{k,1}$, $k = 1, \dots, n$ in involution.

Proof. Obviously after the complexification each of the integrals I_k becomes complex-valued $I_k^{\mathbb{C}} = I_{k,0} + iI_{k,1}$. Since I_k is a real analytic function then $I_k^{\mathbb{C}}$ satisfies Cauchy-Riemann equations with respect to each of the complexified variables. Keeping this in mind let us complexify the dynamical variables in (13). Separating the real and the imaginary parts and making use of Cauchy-Riemann equations we get by direct calculation that:

$$\{I_{k,0}, I_{j,0}\}_0 = \{I_{k,0}, I_{j,1}\}_0 = \{I_{k,1}, I_{j,1}\}_0 = 0, \qquad (14)$$

where

$$\{F, G\}_0 \equiv \sum_{s=1}^{n} \left(\frac{\partial F}{\partial p_{s,0}} \frac{\partial G}{\partial q_{s,0}} - \frac{\partial F}{\partial q_{s,0}} \frac{\partial G}{\partial p_{s,0}} - \right. \qquad (15)$$
$$\left. - \frac{\partial F}{\partial p_{s,1}} \frac{\partial G}{\partial q_{s,1}} + \frac{\partial F}{\partial q_{s,1}} \frac{\partial G}{\partial p_{s,1}} \right) = 0,$$

are the Poisson brackets related to the symplectic form ω_0 (10).

Quite analogously we can prove that this set of integrals are in involution also with respect to the Poisson brackets $\{\cdot, \cdot\}_1$ related to the symplectic form ω_1. Thus we have proved that $\left\{\mathcal{M}^{\mathbb{C}}, H_0, \omega_0\right\}$ has $2n$ integrals in involution.

The next step is to prove that these $2n$ integrals are functionally independent provided the initial ones I_k are. The independence of I_k can be expressed by

$$dI_1 \wedge \cdots \wedge dI_n \equiv \sum_{i_1 < \cdots < i_n} W_{i_1, \dots, i_n} dz_{i_1} \wedge \cdots \wedge dz_{i_n} \neq 0, \quad (16)$$

where $W_{i_1,\dots,i_n}$ is the minor of order n of the $n \times 2n$ matrix W with matrix elements

$$W_{jk} = \frac{\partial I_j}{\partial z_k}, \qquad j = 1, \dots, n, \quad k = 1, \dots, 2n, \qquad (17)$$

determined by the columns $1 \leq i_i < \cdots < i_n \leq 2n$. Here and below we will denote by z the $2n$-component vector with components $z_i = q_i$ and $z_{i+n} = p_i$ for $i = 1, \dots, n$. In other words the independence of I_k means that $\operatorname{rank} W = n$.

In the complexified case both the integrals $\mathcal{I}_k \equiv I_k^{\mathbb{C}}$ and the dynamical variables z_j become complex-valued. Their independence can be formulated by:

$$d\mathcal{I}_1 \wedge \cdots \wedge d\mathcal{I}_n \wedge d\mathcal{I}_1^* \wedge \cdots \wedge d\mathcal{I}_n^* =$$
$$\sum_{k=1}^{2n} \sum_{\substack{i_1 < \cdots < i_k \\ j_{k+1} < \cdots < j_{2n-k}}} W_{i_1,\dots,i_k,j_{k+1},\dots,j_{2n-k}} dz_{i_1} \wedge \cdots \wedge dz_{i_k} \wedge$$
$$\wedge\, dz^*_{j_{k+1}} \wedge \cdots \wedge dz^*_{j_{2n-k}} \neq 0, \quad (18)$$

where

$$\mathcal{W} = \begin{pmatrix} W^{11} & W^{12} \\ W^{21} & W^{22} \end{pmatrix}, \qquad (19)$$

$$W^{11}_{jk} = \frac{\partial \mathcal{I}_j}{\partial z_k}, \qquad W^{22}_{jk} = \frac{\partial \mathcal{I}^*_j}{\partial z^*_k} = (W^{11}_{jk})^*,$$
$$W^{12}_{jk} = \frac{\partial \mathcal{I}_j}{\partial z^*_k}, \qquad W^{21}_{jk} = \frac{\partial \mathcal{I}^*_j}{\partial z_k} = (W^{12}_{jk})^*.$$

Therefore we have to prove now that $\operatorname{rank} \mathcal{W} = 2n$. But due to the analyticity of all I_k the matrix elements of W^{12} and W^{21} vanish and $\mathcal{W}$ becomes a block-diagonal matrix each block being of rank n. The proposition is proved.

Remark 2. If the transition to action-angle variables is not analytic then Proposition 1 is not directly applicable.

4 Hamiltonian reductions and real Hamiltonian forms

In this section we will show how we associate with each Hamiltonian system $\mathcal{H}$ a *family of RHF*. We will do this by using a special type of reductions.

There are several methods to reduce Hamiltonian systems to systems with lower number of degrees of freedom. One of the best known is the reduction by using integrals of motion [8]. Indeed, if we constrain our dynamical system by fixing the values of p integrals of motion $I_j(q_k, p_k) = \text{const}$, $j = 1, \dots, p$, then such constraint is compatible with the dynamics and leads to reduced system with $n - p$ degrees of freedom. Here, we will follow somewhat different route by restricting the dynamics to $\mathcal{M}_{\mathbb{R}}$ which is the fixed point set of a Cartan-like involutive automorphism $\widetilde{\mathcal{C}}$, defined below.

The approach we will follow in this paragraph is inspired by the basic idea of construction of real forms for simple Lie algebras [9]. A basic tool in our construction is the automorphism $\widetilde{\mathcal{C}}$, which plays the rôle of a "complex conjugation operator".

Let us introduce involutive canonical automorphism $\mathcal{C}$ (i.e. $\mathcal{C}^2 = 1$, $\mathcal{C}^*\omega = \omega$) on the phase space $\mathcal{M}^{(n)}$ and on its dual[1] by:

$$\mathcal{C}\left(\{F, G\}\right) = \{\mathcal{C}(F), \mathcal{C}(G)\}, \qquad \mathcal{C}^2 = \mathbb{1}, \qquad (20)$$

[1] In general we should have different notations of $\mathcal{C}$ in these spaces [8]. However, since our phase space is a vector space with some abuse of notations we will use the same letter for both realizations.

where F, $G \in \mathcal{F}(\mathcal{M}^{(n)})$ are real analytic functions on $\mathcal{M}^{(n)}$. The involution acts on them by:

$$\begin{aligned}&\mathcal{C}(F(p_1,\dots,p_n,q_1,\dots,q_n))\\&= F((\mathcal{C}(p_1),\dots,\mathcal{C}(p_n),\mathcal{C}(q_1),\dots,\mathcal{C}(q_n))).\end{aligned} \tag{21}$$

In terms of vector fields $X, Y \in T\mathcal{M}$ and the lifted involution $T\mathcal{C} : T\mathcal{M} \to T\mathcal{M}$ we have:

$$\omega(T\mathcal{C}(X), T\mathcal{C}(Y)) = \mathcal{C}(\omega(X,Y)), \quad (T\mathcal{C})^2 = \mathbb{1} \tag{22}$$

where ω is the symplectic form corresponding to the Poisson brackets in (20).

Since $\mathcal{C}$ has eigenvalues 1 and -1, it naturally splits $\mathcal{M}^{(n)}$ into two subspaces $\mathcal{M}^{(n)} = \mathcal{M}_+^{(n_+)} \oplus \mathcal{M}_-^{(n_-)}$ such that

$$\begin{aligned}\mathcal{C}\mathbf{X} &= \mathbf{X} \quad \text{for} \quad \mathbf{X} \in \mathcal{M}_+^{(n_+)},\\ \mathcal{C}\mathbf{Y} &= -\mathbf{Y} \quad \text{for} \quad \mathbf{Y} \in \mathcal{M}_-^{(n_-)}\end{aligned} \tag{23}$$

where $n_\pm = \dim \mathcal{M}_\pm^{(n_\pm)}$, $n = n_+ + n_-$. We also assume that the starting Hamiltonian H is invariant with respect to $\mathcal{C}$:

$$\mathcal{C}(H) = H. \tag{24}$$

Equation (20) guarantees that each of the subspaces $\mathcal{M}_+^{(n_+)}$, $\mathcal{M}_-^{(n_-)}$ is a symplectic subspace of $\mathcal{M}_\mathbb{C}^{(2n)}$. On $\mathcal{M}_+^{(n_+)}$ and $\mathcal{M}_-^{(n_-)}$ we have

$$\omega_+ = \sum_{k=1}^{n_+} dp_k^+ \wedge dq_k^+. \qquad \omega_- = \sum_{k=1}^{n_-} dp_k^- \wedge dq_k^-$$

where p_k^+, q_k^+ (resp. p_k^-, q_k^-) are basis in $\mathcal{M}_+^{(n_+)}$ (resp. $\mathcal{M}_-^{(n_-)}$). With respect to the automorphism $\mathcal{C}$ they satisfy:

$$\mathcal{C}(p_k^\pm) = \pm p_k^\pm, \qquad \mathcal{C}(q_k^\pm) = \pm q_k^\pm \tag{25}$$

for all $k = 1, \dots, n_\pm$.

In $\mathcal{M}_\mathbb{C}^{(n)}$ along with $\mathcal{C}$ we can introduce also the complex conjugation $*$. In this construction obviously $\mathcal{C}$ commutes with $*$ and their composition $\widetilde{\mathcal{C}} = \mathcal{C} \circ * = * \circ \mathcal{C}$ is again an involutive automorphism on $\mathcal{M}_\mathbb{C}^{(2n)}$.

Corollary 1. *Let I_j, $j = 1, \dots, n$ be n integrals of motion in involution of our initial system $\mathcal{H}$, depending analytically on q_k and p_k. Let us denote their extensions to $\mathcal{M}^\mathbb{C}$ by $\mathcal{I}_j = I_{j,0} + iI_{j,1}$. Then both $\mathcal{I}_j$ and*

$$\widetilde{\mathcal{C}}(\mathcal{I}_j) = \mathcal{I}_j^* \tag{26}$$

are also integrals of motion in involution for the complexified system $\mathcal{H}^\mathbb{C}$.

Using equation (20) it is easy to check that the dynamics generated by a Hamiltonian which is invariant with respect to $\mathcal{C}$ (24) will have the subspaces $\mathcal{F}_+$ and $\mathcal{F}_-$ of $\mathcal{C}$-invariant and $\mathcal{C}$-antiinvariant functions as invariant subspaces. Moreover $\mathcal{F}_+$ turns out to be also an invariant subalgebra.

The real form $\mathcal{M}_\mathbb{R}^{(n)}$ of the phase space is the symplectic (with respect to ω_0) subspace of $\mathcal{M}_\mathbb{C}^{(2n)}$ invariant with respect to $\widetilde{\mathcal{C}}$:

$$\mathcal{M}_\mathbb{R}^{(n)} = \mathcal{M}_+^{(n_+)} \oplus i\mathcal{M}_-^{(n_-)}. \tag{27}$$

Indeed any element of $\mathcal{M}_\mathbb{R}^{(n)}$ can be represented as:

$$\boldsymbol{Z} = \boldsymbol{X} + i\boldsymbol{Y} \in \mathcal{M}_\mathbb{R}^{(n)}. \tag{28}$$

where $\boldsymbol{X}$ and $\boldsymbol{Y}$ are real-valued elements of $\mathcal{M}_+^{(n_+)}$ and $\mathcal{M}_-^{(n_-)}$ respectively. The reality condition means that

$$\tilde{\mathcal{C}}(\boldsymbol{Z}) \equiv \mathcal{C}(\boldsymbol{Z}^*) = \mathcal{C}(\boldsymbol{X} - i\boldsymbol{Y}) = \boldsymbol{X} + i\boldsymbol{Y} = \boldsymbol{Z} \tag{29}$$

where we have made use of equation (23).

Here and below by $F_\mathbb{R}$ and $G_\mathbb{R} \in \mathcal{F}(\mathcal{M}_\mathbb{R}^{(n)})$ we denote the restriction of the observables F, G by restricting their arguments to $\mathcal{M}_\mathbb{R}$. Then $F_\mathbb{R}$ and $G_\mathbb{R}$ satisfy the analog of equation (21) with $\mathcal{C}$ replaced by $\widetilde{\mathcal{C}}$. Due to equation (20) their Poisson bracket $\{F_\mathbb{R}, G_\mathbb{R}\} \in \mathcal{F}(\mathcal{M}_\mathbb{R}^{(n)})$ too, i.e. $\mathcal{F}(\mathcal{M}_\mathbb{R}^{(n)})$ becomes a Poisson subalgebra. If we choose the Hamiltonian $H_\mathbb{R} \in \mathcal{F}(\mathcal{M}_\mathbb{R}^{(n)})$ then

$$\widetilde{\mathcal{C}}(H_\mathbb{R}) = H_\mathbb{R}. \tag{30}$$

The evolution associated with $H_\mathbb{R}$ of the dynamical variable $F_\mathbb{R}$:

$$\frac{dF_\mathbb{R}}{dt} = \{H_\mathbb{R}, F_\mathbb{R}\}, \tag{31}$$

defines a dynamics on $\mathcal{M}_\mathbb{R}^{(n)}$. Rewriting (31) into its equivalent form:

$$\omega(X_{H_\mathbb{R}}, \cdot) = dH_\mathbb{R} \tag{32}$$

and making use of (22) we see that the vector field $X_{H_\mathbb{R}}$ must also satisfy

$$\widetilde{C}(X_{H_\mathbb{R}}) = X_{\widetilde{C}(H_\mathbb{R})} = X_{H_\mathbb{R}} \tag{33}$$

on $\mathcal{M}_\mathbb{R}^{(n)}$. The symplectic form restricted on $\mathcal{M}_\mathbb{R}^{(n)}$ also becomes real and equals

$$\omega_\mathbb{R} = \sum_{k=1}^{n_+} dp_{k,0}^+ \wedge dq_{k,0}^+ - \sum_{k=1}^{n_-} dp_{k,1}^- \wedge dq_{k,1}^- \tag{34}$$

where $p_{k,0}^+, q_{k,0}^+, k = 1, \dots, n_+$ and $p_{k,1}^-, q_{k,1}^-, k = 1, \dots, n_-$ are the basic elements in $\mathcal{M}_\mathbb{R}^{(n)}$.

If $\mathcal{M}_\mathbb{C}^{(2n)}$ is endowed with Hamiltonian which is "real" with respect to $\widetilde{\mathcal{C}}$ and whose vector field X_H satisfies (33) then the restriction of the dynamics on $\{\mathcal{M}_\mathbb{R}^{(n)}, \omega_\mathbb{R}, H_\mathbb{R}\}$ is well defined and coincides with the dynamics on $\{\mathcal{M}_\mathbb{C}^{(2n)}, \omega_0, H_0\}$ restricted to $\mathcal{M}_\mathbb{R}^{(n)}$.

The complexified equations of motion take the form:

$$\begin{aligned} \frac{dp^+_{k,0}}{dt} &= -\frac{\partial H_0}{\partial q^+_{k,0}}, & \frac{dp^+_{k,1}}{dt} &= \frac{\partial H_0}{\partial q^+_{k,1}}, \\ \frac{dp^-_{k,1}}{dt} &= \frac{\partial H_0}{\partial q^-_{k,1}}, & \frac{dp^-_{k,0}}{dt} &= -\frac{\partial H_0}{\partial q^-_{k,0}}, \\ \frac{dq^+_{k,0}}{dt} &= \frac{\partial H_0}{\partial p^+_{k,0}}, & \frac{dq^+_{k,1}}{dt} &= -\frac{\partial H_0}{\partial p^+_{k,1}}, \\ \frac{dq^-_{k,1}}{dt} &= -\frac{\partial H_0}{\partial p^-_{k,1}}, & \frac{dq^-_{k,0}}{dt} &= \frac{\partial H_0}{\partial p^-_{k,0}}. \end{aligned} \tag{35}$$

We shall show that the dynamics on $\{\mathcal{M}_{\mathbb{C}}^{(2n)}, \omega_0, H_0\}$ reduces naturally to $\mathcal{M}_{\mathbb{R}}$ which is the fixed point set of $\widetilde{\mathcal{C}}$.

Proposition 2. *Let the Hamiltonian system $\mathcal{H}^{\mathbb{C}}$ and the involution $\tilde{\mathcal{C}}$ be as above. Then the second class Dirac constraints*

$$p^+_{k,1} = 0, \qquad q^+_{k,1} = 0, \qquad p^-_{k,0} = 0, \qquad q^-_{k,0} = 0 \tag{36}$$

on the subspace $\mathcal{M}_{\mathbb{R}}$ invariant with respect to $\tilde{\mathcal{C}}$ are compatible with the dynamics of the complexified Hamiltonian system $\mathcal{H}^{\mathbb{C}}$. The reduced equations of motion take the form:

$$\begin{aligned} \frac{dp^+_{k,0}}{dt} &= -\frac{\partial H_{\mathbb{R}}}{\partial q^+_{k,0}}, \\ \frac{dp^-_{k,1}}{dt} &= \frac{\partial H_{\mathbb{R}}}{\partial q^-_{k,1}}, \\ \frac{dq^+_{k,0}}{dt} &= \frac{\partial H_{\mathbb{R}}}{\partial p^+_{k,0}}, \\ \frac{dq^-_{k,1}}{dt} &= -\frac{\partial H_{\mathbb{R}}}{\partial p^-_{k,1}} \end{aligned} \tag{37}$$

where $H_{\mathbb{R}} = H_0|_{\mathcal{M}_{\mathbb{R}}}$.

Proof. Since H is real analytic function, then

$$H_0(\dots, p^+_{k,1}, q^+_{k,1}, \dots) = H_0(\dots, -p^+_{k,1}, -q^+_{k,1}, \dots)$$

is an even function of $p^+_{k,1}$ and $q^+_{k,1}$. Besides from the condition $C(H) = H$ it follows, that

$$H_0(\dots, p^-_{k,0}, q^-_{k,0}, \dots) = H_0(\dots, -p^-_{k,0}, -q^-_{k,0}, \dots)$$

and H_0 is an even function also of $p^-_{k,0}$ and $q^-_{k,0}$. Obviously the first derivatives of H_0 with respect to these variables are odd functions and therefore:

$$\left.\frac{\partial H_0}{\partial q^+_{k,1}}\right|_{\mathcal{M}_{\mathbb{R}}} = \left.\frac{\partial H_0}{\partial q^-_{k,0}}\right|_{\mathcal{M}_{\mathbb{R}}} = \left.\frac{\partial H_0}{\partial p^+_{k,1}}\right|_{\mathcal{M}_{\mathbb{R}}} = \left.\frac{\partial H_0}{\partial p^-_{k,0}}\right|_{\mathcal{M}_{\mathbb{R}}} = 0. \tag{38}$$

The real Hamiltonian form is determined by:

$$\mathcal{H}_{\mathbb{R}} \equiv \{\mathcal{M}_{\mathbb{R}}, H_{\mathbb{R}}(p,q), \omega_{\mathbb{R}}\},$$

$$\mathcal{M}_{\mathbb{R}} = \mathcal{M}_+ \oplus i\mathcal{M}_-, \qquad H_{\mathbb{R}}(p,q) = H_0(p_k, q_k)|_{\mathcal{M}_{\mathbb{R}}},$$

$$\omega_{\mathbb{R}} = \omega_0|_{\mathcal{M}_{\mathbb{R}}} = \sum_{k=1}^{n_+} dp^+_{k,0} \wedge dq^+_{k,0} - \sum_{k=1}^{n_-} d\tilde{p}^-_{k,1} \wedge d\tilde{q}^-_{k,1}$$

where $d\tilde{p}^-_{k,1} = -idp^-_{k,1}$, $d\tilde{q}^-_{k,1} = -idq^-_{k,1}$. The proposition is proved.

Thus we have proved that the equations (35) can be consistently restricted to $\mathcal{M}_{\mathbb{R}}$ and give rise to a well defined dynamical system with n degrees of freedom $\mathcal{H}_{\mathbb{R}} \equiv \{\mathcal{M}_{\mathbb{R}}, \omega_{\mathbb{R}}, H_{\mathbb{R}}\}$ which we call a real Hamiltonian form (RHF) of the initial Hamiltonian system $\mathcal{H} \equiv \{\mathcal{M}, \omega, H\}$.

Let us now consider the set of observables $F_{\mathbb{R}}$, $G_{\mathbb{R}}$ related to our RHF $\mathcal{H}_{\mathbb{R}}$. They can be obtained from the observables of the complexified system $\mathcal{H}^{\mathbb{C}}$ by restricting their variables to $\mathcal{M}_{\mathbb{R}}$. We remind that we are selecting the class of observables which depends analytically on the dynamical variables. This means, that after restricting on $\mathcal{M}_{\mathbb{R}}$ they satisfy

$$\left.\frac{\partial F_0}{\partial q^+_{k,1}}\right|_{\mathcal{M}_{\mathbb{R}}} = \left.\frac{\partial F_0}{\partial q^-_{k,0}}\right|_{\mathcal{M}_{\mathbb{R}}} = \left.\frac{\partial F_0}{\partial p^+_{k,1}}\right|_{\mathcal{M}_{\mathbb{R}}} = \left.\frac{\partial F_0}{\partial p^-_{k,0}}\right|_{\mathcal{M}_{\mathbb{R}}} = 0 \tag{39}$$

and analogous relations for the partial derivatives of G. If we now calculate the Poisson brackets (15) between two observables $F_{\mathbb{R}}$ and $G_{\mathbb{R}}$ we easily find that due to equations (39) they will simplify to

$$\begin{aligned} \{F_{\mathbb{R}}, G_{\mathbb{R}}\}_0 \equiv& \sum_{s=1}^{n_+} \left(\frac{\partial F_{\mathbb{R}}}{\partial p^+_{s,0}} \frac{\partial G_{\mathbb{R}}}{\partial q^+_{s,0}} - \frac{\partial F_{\mathbb{R}}}{\partial q^+_{s,0}} \frac{\partial G_{\mathbb{R}}}{\partial p^+_{s,0}} \right) \\ &- \sum_{s=n_++1}^{n} \left(\frac{\partial F_{\mathbb{R}}}{\partial p^-_{s,1}} \frac{\partial G_{\mathbb{R}}}{\partial q^-_{s,1}} - \frac{\partial F_{\mathbb{R}}}{\partial q^-_{s,1}} \frac{\partial G_{\mathbb{R}}}{\partial p^-_{s,1}} \right) \end{aligned} \tag{40}$$

Obviously the Poisson brackets (40) correspond to the symplectic form $\omega_{\mathbb{R}}$.

Proposition 3. *The RHF $\mathcal{H}_{\mathbb{R}}$ corresponding to a Liouville integrable Hamiltonian system $\mathcal{H}$ is Liouville integrable.*

Proof. We start with a $\mathcal{H}$ which has n integrals in involution I_k depending analytically on the dynamical variables. The complexification provides us with $2n$ integrals of motion $\mathcal{I}_k$ and $\mathcal{I}_k^*$ which are also in involution. Let us now restrict ourselves to $\mathcal{M}_{\mathbb{R}}$. To this end we use the involution $\widetilde{\mathcal{C}}$. It is easy to check that n of the integrals are preserved:

$$\mathcal{I}^+_{k,\mathbb{R}} = \frac{1}{2}\left.\left(\mathcal{I}_k + \widetilde{\mathcal{C}}(\mathcal{I}_k)\right)\right|_{\mathcal{M}_{\mathbb{R}}} \tag{41}$$

and become invariant with respect to $\widetilde{\mathcal{C}}$ while the other n integrals vanish:

$$\mathcal{I}^-_{k,\mathbb{R}} = \frac{1}{2}\left.\left(\mathcal{I}_k - \widetilde{\mathcal{C}}(\mathcal{I}_k)\right)\right|_{\mathcal{M}_{\mathbb{R}}} = 0, \tag{42}$$

The fact that the integrals $\mathcal{I}^+_{k,\mathbb{R}}$ are in involution with respect to the Poisson brackets $\{\cdot,\cdot\}_{\mathbb{R}}$ follows from (40). The proposition is proved.

Those I_j which have definite $\mathcal{C}$-parity will produce real first integrals $I_{j,\mathbb{R}}$ for the real form dynamics and those with minus $\mathcal{C}$-parity will produce purely imaginary integrals.

Remark 3. We proved that after restricting $\mathcal{H}^{\mathbb{C}}$ to $\mathcal{M}_{\mathbb{R}}$ the integrals of motion satisfy (42). One may ask whether the inverse statement also holds true. Namely, assume that we restrict $\mathcal{H}^{\mathbb{C}}$ by using simultaneously all n constraints in (42) together with their symplectic conjugate ones. Since $\mathcal{I}^-_{k,\mathbb{R}}$ are all independent on $\mathcal{M}_{\mathbb{C}}$ then such a procedure would lead to a dynamical system with n degrees of freedom. Will this new system coincide with $\mathcal{H}_{\mathbb{R}}$? We believe that the answer to this question is positive, though we do not yet have a rigorous proof of this fact.

If we restrict our dynamical variables on $\mathcal{M}_{\mathbb{R}}$ then only the integrals invariant with respect to $\widetilde{\mathcal{C}}$ survive.

Let us assume now that we have the complexified system $\mathcal{H}^{\mathbb{C}}$ restricted by (42). Let us now pick a point $m \in \mathcal{M}_{\mathbb{R}}$ which will correspond to the initial condition of our dynamical equations. Obviously the evolution can not take this point out of $\mathcal{M}_{\mathbb{R}}$. Indeed, if we assume the opposite we easily find that the corresponding Hamiltonian vector field is not invariant with respect to $\widetilde{\mathcal{C}}$. This means that the Hamiltonian H is not invariant with respect to $\mathcal{C}$ which is a contradiction.

Following the same line we can prove that if the initial point belongs to $\mathcal{M}\backslash\mathcal{M}_{\mathbb{R}}$ then the evolution can not take it out of $\mathcal{M}\backslash\mathcal{M}_{\mathbb{R}}$.

As a result the set of constraints (42) split $\mathcal{M}_{\mathbb{C}}$ into two orthogonal invariant subspaces.

For some special choices of H it may happen that the dynamics of $\mathcal{H}$ and $\mathcal{H}_{\mathbb{R}}$ coincide.

Despite the apparent simplicity of the transition to the real form dynamics it is essentially different from the initial one—the real form dynamical vector field satisfying $\Gamma_{\mathbb{R}} \lrcorner\, \omega_{\mathbb{R}} = -dH_{\mathbb{R}}$ is generally not even a (locally) Hamiltonian vector field for the initial ω. As a result we may expect that the new and the initial dynamical vector fields will coincide only for very few cases. Conditions for this are settled in the following

Proposition 4. *$\Gamma_{\mathbb{R}} = \Gamma$ iff*

i) the Hamiltonian is separable in a sum of two parts depending on as follows: $H = H_+(p_k^+, q_k^+) + H_-(p_j^-, q_j^-)$ where q_k^+, p_k^+ (resp. q_j^-, p_j^-) are elements of $\mathcal{M}_+$ (resp. $\mathcal{M}_-$), and

ii) $H_-(ip_j^-, iq_j^-) = -H_-(p_j^-, q_j^-)$.

Also, condition i) is equivalent both to the local Hamiltonianity of Γ with respect to $\omega_{\mathbb{R}}$ and to the compatibility of Γ with $\omega_{\mathbb{R}}$ i.e. $\mathcal{L}_\Gamma \omega_{\mathbb{R}} = 0$.

Proof. The dynamical vector field satisfying $\Gamma_{\mathbb{R}} \lrcorner\, \omega_{\mathbb{R}} = -dH_{\mathbb{R}}$ will be

$$\Gamma_{\mathbb{R}} = \sum_{k=1}^{n_+} \left(\frac{\partial H_{\mathbb{R}}}{\partial p_k^+} \frac{\partial}{\partial q_k^+} - \frac{\partial H_{\mathbb{R}}}{\partial q_k^+} \frac{\partial}{\partial p_k^+} \right) + \sum_{j=1}^{n_-} \left(\frac{\partial H_{\mathbb{R}}}{\partial q_j^-} \frac{\partial}{\partial p_j^-} - \frac{\partial H_{\mathbb{R}}}{\partial p_j^-} \frac{\partial}{\partial q_j^-} \right) \tag{43}$$

and we have to satisfy:

$$\frac{\partial H}{\partial q_k^+} = \frac{\partial H_{\mathbb{R}}}{\partial q_k^+} \qquad \frac{\partial H}{\partial p_k^+} = \frac{\partial H_{\mathbb{R}}}{\partial p_k^+}$$
$$\frac{\partial H}{\partial q_j^-} = -\frac{\partial H_{\mathbb{R}}}{\partial q_j^-} \qquad \frac{\partial H}{\partial p_j^-} = -\frac{\partial H_{\mathbb{R}}}{\partial p_j^-}. \tag{44}$$

Comparing the equations from the first and second rows of (44) we obtain that all mixed derivatives of H should vanish:

$$\frac{\partial^2 H}{\partial q_k^+ \partial q_j^-} = 0, \qquad \frac{\partial^2 H}{\partial p_k^+ \partial p_j^-} = 0,$$
$$\frac{\partial^2 H}{\partial q_k^+ \partial p_j^-} = 0, \qquad \frac{\partial^2 H}{\partial p_k^+ \partial q_j^-} = 0 \tag{45}$$

which is exactly the separability condition *i)* of the Hamiltonian. Now *ii)* part of the Proposition follows trivially from the way we construct $H_{\mathbb{R}}$.

Due to the closedness of ω (and $\omega_{\mathbb{R}}$) we have:

$$\mathcal{L}_\Gamma \omega_{\mathbb{R}} = d(\Gamma \lrcorner\, \omega_{\mathbb{R}}) = 2 \sum_{j,k} \left(\frac{\partial^2 H}{\partial q_k^+ \partial p_j^-} dq_k^+ \wedge dp_j^- + \frac{\partial^2 H}{\partial q_j^- \partial p_k^+} dq_j^- \wedge dp_k^+ \right) \tag{46}$$

and vanishing of all mixed derivatives in equation (45) is the condition for the required compatibility and/or local Hamiltonianity. Since the separability of H is equivalent to the separability of $H_{\mathbb{R}}$ we have similarly compatibility of $\Gamma_{\mathbb{R}}$ with ω and local Hamiltonianity of $\Gamma_{\mathbb{R}}$ with respect to ω.

When the conditions of the Proposition 4 are fulfilled our procedure will give us a bi-Hamiltonian description of the initial dynamics with a recursion operator

$$T = \omega^{-1} \circ \omega_{\mathbb{R}} = \mathbb{1}_+ - \mathbb{1}_-$$

where $\mathbb{1}_\pm$ are the identity tensors on $\mathcal{M}_\pm^{(n_\pm)}$. So, whatever the outcome we gain either new dynamics or a bi-Hamiltonianity of the old one.

For instance, if we have a collection of uncoupled harmonic oscillators $H = \sum_i (p_i^2 + q_i^2)$ and a compatible involution: $\mathcal{C}(p_i) = \epsilon_i p_i$, $\mathcal{C}(q_i) = \epsilon_i q_i$ with $\epsilon_i = \pm 1$ then we will have $H_{\mathbb{R}} = \sum_i \epsilon_i (p_i^2 + q_i^2)$ and $\Gamma_{\mathbb{R}} = \Gamma$ i.e. coinciding initial and real form dynamics. The same will be also true for $U(ip_j^-, iq_j^-) = -U(p_j^-, q_j^-)$.

At present, we do not have a receipt how to construct and classify all involutions of $\mathcal{M}$ consistent with a given Hamiltonian. However in many important cases it is possible to provide a number of such non-trivial involutions.

5 RHF of completely integrable systems

Here we understand the notion of completely integrable systems in a broader sense [10]. We will say that a dynamical system with n degrees of freedom is completely integrable if we find $2n$ functions $\{I_k, \phi_k\}$, $k = 1, \dots, n$ such that I_k are integrals of motion, i.e.:

$$L_\Gamma I_k = 0, \qquad k = 1, \dots, n, \tag{47}$$

and ϕ_k are nilpotent with respect to the vector field L_Γ of order 2:

$$L_\Gamma L_\Gamma \phi_k = 0, \qquad k = 1, \dots, n. \tag{48}$$

If there is one or more functions of the ϕ's which are constants of the motion, the system is said to be superintegrable [11]. We also assume that $\{I_k, \phi_k\}$ introduce local coordinates on our phase space.

Such set of functions $\{I_k, \phi_k\}$ generalize the standard notion of action-angle variables (AAV) [10]. While the standard AAV can be introduced only for compact motion on a torus, our variables $\{I_k, \phi_k\}$ can be derived also for non-compact dynamics. This is important to our purposes, especially when we analyze how the transition from one RHF to another changes the character of the motion from compact to a non-compact one and vice versa, see Section 8 below. Therefore from now on, with some abuse of language we will say that $\{I_k, \phi_k\}$ are our AAV.

The complexification renders all $I_k^\pm$ and $\phi_k^\pm$ complex: $I_k^{\mathbb{C},\pm} = I_{k,0}^\pm + iI_{k,1}^\pm$ and $\phi_k^{\mathbb{C},\pm} = \phi_{k,0}^\pm + i\phi_{k,1}^\pm$ and the automorphism $\widetilde{\mathcal{C}}$ has the form:

$$\widetilde{\mathcal{C}}(dI_k^{\mathbb{C},\pm}) = \pm(dI_k^{\mathbb{C},\pm})^*, \qquad \widetilde{\mathcal{C}}(d\phi_k^{\mathbb{C},\pm}) = \pm(d\phi_k^{\mathbb{C},\pm})^* \tag{49}$$

The automorphism $\widetilde{\mathcal{C}}$ obviously satisfies the condition (33). In order to satisfy also (24) we need to assume that H is an even function of all $I_k^- \in \mathcal{M}_-$. Then the restriction on $\mathcal{M}_\mathbb{R}$ according to (28) and (49) means restricting all $I_k^+, \phi_k^+ \in \mathcal{M}_+$ to be real, while all $I_k^-, \phi_k^- \in \mathcal{M}_-$ become purely imaginary. Then:

$$\begin{aligned} H_\mathbb{R} &= H(I_{1,0}^+, \dots, I_{n_+,0}^+, iI_{1,1}^-, \dots, iI_{n_-,1}^-), \\ \omega_\mathbb{R} &= \sum_{k=1}^{n_+} dI_{k,0}^+ \wedge d\phi_{k,0}^+ - \sum_{k=1}^{n_-} dI_{k,1}^- \wedge d\phi_{k,1}^- \end{aligned} \tag{50}$$

and obviously we have again a completely integrable Hamiltonian system.

Till the rest of this section we also assume that the Hamiltonian is separable, i.e.

$$H = \sum_{k=1}^{n_+} h_k^+(I_k^+) + \sum_{k=1}^{n_-} h_k^-(I_k^-). \tag{51}$$

Obviously the condition (24) requires that $h_k^-(I_k^-)$ must be even functions of I_k^-.

Another important approach to completely integrable systems [12] is based on the notion of the recursion operator – a (1,1) tensor field with vanishing Nijenhuis torsion [13]. This tensor field reflects the possibility to introduce a second symplectic structure ω_1 on $\mathcal{M}$ through:

$$\omega_1(X, Y) = \frac{1}{2}\left(\omega(TX, Y) + \omega(X, TY)\right). \tag{52}$$

In terms of $I_k^\pm$, $\phi_k^\pm$ the tensor field T and ω_1 can be expressed by:

$$\begin{aligned} T &= \sum_{k=1}^{n_+} T_k^+ + \sum_{k=1}^{n_-} T_k^-, \qquad \omega_1 = \sum_{k=1}^{n_+} \omega_{1,k}^+ + \sum_{k=1}^{n_-} \omega_{1,k}^-, \\ T_k^\pm &= \zeta_k^\pm(I_k^\pm)\left(dI_k^\pm \otimes \frac{\partial}{\partial I_k^\pm} + d\phi_k^\pm \otimes \frac{\partial}{\partial \phi_k^\pm}\right), \\ \omega_{1,k}^\pm &= \zeta_k^\pm(I_k^\pm) dI_k^\pm \wedge d\phi_k^\pm \end{aligned} \tag{53}$$

where $\zeta_k^\pm(I_k^\pm)$ are some functions of $I_k^\pm$; we assume that they are real analytic functions of their variables.

In order that ω_1 also satisfy equation (22) it is enough that $\mathcal{C}(T) = T$. In particular this means that ζ_k^- must be even functions of I_k^-. If this is so then we can repeat our construction also for ω_1; i.e., we can complexify it and then restrict it onto $\mathcal{M}_\mathbb{R}$ with the result:

$$\begin{aligned} T_\mathbb{R} &= \sum_{k=1}^{n_+} T_{\mathbb{R},k}^+ + \sum_{k=1}^{n_-} T_{\mathbb{R},k}^-, \\ T_{\mathbb{R},k}^+ &= \zeta_k^+(I_k^+)\left(dI_k^+ \otimes \frac{\partial}{\partial I_k^+} + d\phi_k^+ \otimes \frac{\partial}{\partial \phi_k^+}\right), \\ T_{\mathbb{R},k}^- &= \zeta_k^-(iI_k^-)\left(dI_k^- \otimes \frac{\partial}{\partial I_k^-} + d\phi_k^- \otimes \frac{\partial}{\partial \phi_k^-}\right), \\ \omega_{1,\mathbb{R}} &= \sum_{k=1}^{n_+} \zeta_k^+(I_k^+) dI_k^+ \wedge d\phi_k^+ - \sum_{k=1}^{n_-} \zeta_k^-(iI_k^-) dI_k^- \wedge d\phi_k^-. \end{aligned} \tag{54}$$

Thus by construction, the restriction to other real forms preserves the Nijenhuis property of T and the condition for double degenerate and nowhere constant eigenvalues. Also it is easy to check that it is preserved by the dynamical flow: $\mathcal{L}_{\Gamma_\mathbb{R}} T_\mathbb{R} = 0$. Due to the fact that existence of recursion operators is equivalent [14] to integrability at least in the non-resonant case, this line of argumentation gives us another instrument to treat the integrability of real forms.

The separability of H (51) looks rather restrictive condition. However, all integrable systems obtained by reducing a soliton equation (like, e.g. the nonlinear Schrödinger equation) on its N-soliton sector are separable. Finite-dimensional systems allowing Lax pairs like Toda chains, Calogero-Moser systems etc. also possess this property, see equation (70) below.

6 Examples

We illustrate our construction by several paradigmatic examples which include several types of Toda chain models and Calogero-Moser models.

Example 1. Toda chain related to the $sl(n,\mathbb{C})$ algebra. We consider simultaneously the conformal and the affine case by introducing the parameter c_0 which takes two values: $c_0 = 0$ (conformal case) and $c_0 = 1$ (affine case)

$$H_{\mathrm{TC}} = \sum_{k=1}^{n} \frac{p_k^2}{2} + \sum_{k=1}^{n-1} e^{q_{k+1}-q_k} + c_0 e^{q_1 - q_n},$$
$$\omega = \sum_{k=1}^{n} dp_k \wedge dq_k. \tag{55}$$

We choose the involution as:

$$\mathcal{C}(p_k) = -p_{\bar{k}}, \qquad \mathcal{C}(q_k) = -q_{\bar{k}} \tag{56}$$

where $\bar{k} = n + 1 - k$. We also introduce new symplectic coordinates adapted to the action of $\mathcal{C}$:

$$p_k^+ = \frac{p_k - p_{\bar{k}}}{\sqrt{2}}, \qquad q_k^+ = \frac{q_k - q_{\bar{k}}}{\sqrt{2}}, \tag{57}$$
$$\widetilde{p_k^-} = \frac{p_k + p_{\bar{k}}}{\sqrt{2}}, \qquad \widetilde{q_k^-} = \frac{q_k + q_{\bar{k}}}{\sqrt{2}}, \qquad k = 1, \ldots, r = [n/2]$$

where $n = 2r + r_0$ with $r_0 = 0$ or 1; for $r_0 = 1$ we have to add also: $\widetilde{p} = p_{r+1}$, $\widetilde{q} = q_{r+1}$. These coordinates are such that

$$\mathcal{C}(p_k^{\pm}) = \pm p_k^{\pm}, \qquad \mathcal{C}(\widetilde{q}_k^{\pm}) = \pm \widetilde{q}_k^{\pm}, \tag{58}$$
$$\mathcal{C}(\widetilde{p}) = -\widetilde{p}, \qquad \mathcal{C}(\widetilde{q}) = -\widetilde{q}.$$

As a result we obtain the following real forms of the TC model: i) for $n = 2r + 1$:

$$\begin{aligned} H_{\mathrm{TC1}} &= \frac{1}{2}\sum_{k=1}^{r}((p_k^+)^2 - (p_k^-)^2) - \frac{1}{2}(p_{r+1}^-)^2 \\ &\quad + 2\sum_{k=1}^{r-1} e^{(q_{k+1}^+ - q_k^+)/\sqrt{2}} \cos\frac{q_{k+1}^- - q_k^-}{\sqrt{2}} \\ &\quad + 2e^{-q_r^+/\sqrt{2}} \cos\left(q_{r+1}^- - \frac{q_r^-}{\sqrt{2}}\right), \end{aligned} \tag{59}$$
$$\omega_{\mathbb{R}} = \sum_{k=1}^{r} dp_k^+ \wedge dq_k^+ - \sum_{k=1}^{r+1} dp_k^- \wedge dq_k^-.$$

ii) for $n = 2r$:

$$H_{\mathrm{TC2}} = \frac{1}{2}\sum_{k=1}^{r}((p_k^+)^2 - (p_k^-)^2) + e^{-\sqrt{2}q_r^+} \tag{60}$$
$$+ 2\sum_{k=1}^{r-1} e^{(q_{k+1}^+ - q_k^+)/\sqrt{2}} \cos\frac{q_{k+1}^- - q_k^-}{\sqrt{2}},$$
$$\omega_{\mathbb{R}} = \sum_{k=1}^{r} dp_k^+ \wedge dq_k^+ - \sum_{k=1}^{r} dp_k^- \wedge dq_k^-.$$

These models are generalizations of the well known Toda chain models associated to the classical Lie algebras [15]; indeed if we put $q_k^- \equiv 0$ and $p_k^- \equiv 0$ we find that (59) goes into the $\mathbf{B}_r$ TC while (60) provides the $\mathbf{C}_r$ TC.

Example 2. The real Hamiltonian forms for the Calogero–Moser systems (CMS) [16]. The CMS corresponding to the root systems of A, B, C, D and BC-series [17] are defined by Hamiltonians of the type:

$$H_{\mathrm{CMS}} = 1/2 \sum_{j=1}^{n} p_j^2 + U(q) \tag{61}$$

with:

$$U = \begin{cases} g^2 V_-^n, & \text{for} \quad A_{n-1} \\ g^2(V_-^n + V_+^n) + g_1^2 V_1^n + g_2^2 V_2^n, & \text{for} \quad BC_n. \end{cases}$$

Here

$$V_\pm^n = \sum_{j<k\leq n} v(q_j \pm q_k), \qquad V_1^n = \sum_{j\leq n} v(q_j),$$
$$V_2^n = \sum_{j\leq n} v(2q_j), \tag{62}$$

and the function $v(x)$ is one of the following:

$$v(q) = \frac{1}{q^2}, \quad \frac{1}{\sin^2 q}, \quad \frac{1}{\sinh^2 q}, \quad \frac{1}{q^2} + \omega q^2, \quad \wp(q) \tag{63}$$

where $\wp(q)$ is the Weierstrass function.

The BC_n case contains in itself the CMS for the algebras B_n, C_n and D_n; they are obtained by putting $g_2 = 0$, $g_1 = 0$ and $g_1 = g_2 = 0$ respectively.

All these models are invariant under the involution (56). For convenience we again use the coordinates (58) together with the following complex coordinates

$$z_j = \frac{1}{\sqrt{2}}\left(q_k^+ + iq_k^-\right), \qquad z_j^* = \frac{1}{\sqrt{2}}\left(q_k^+ - iq_k^-\right),$$

$k = 1, \ldots, r$.

In order to obtain the Hamiltonian of the RHF of the CMS we have to express H_{CMS} (61) in the new coordinates and assume that $\widetilde{p}_k^-$, $\widetilde{q}_k^-$, $\widetilde{p}$ and $\widetilde{q}$ are purely imaginary and then calculate its real part. We shall denote the imaginary parts of these variables by the same letter but without tilde and so all tilde-less variables will be real. As a result:

$$\mathrm{Re}\sum_{j=1}^{n}\frac{p_j^2}{2} = \frac{1}{2}\sum_{k=1}^{r}\left(p_k^+\right)^2 - \frac{1}{2}\sum_{k=1}^{r}\left(p_k^-\right)^2 - \frac{1}{2}r_0 p^2,$$
$$\mathrm{Re}\, V_1^n = 2\mathrm{Re}\sum_{k=1}^{r} v(z_j) - r_0 v(q),$$
$$\mathrm{Re}\, V_2^n = 2\mathrm{Re}\sum_{k=1}^{r} v(2z_j) - r_0 v(2q),$$
$$\begin{aligned}\mathrm{Re}\, V_-^n &= 2\mathrm{Re}\sum_{j<k\leq r}\left[v(z_j - z_k) + v(z_j + z_k^*)\right] \\ &\quad + \sum_{k=1}^{r} v(\sqrt{2}q_k^+) + 2r_0\mathrm{Re}\sum_{j\leq r} v(z_j - iq),\end{aligned}$$
$$\begin{aligned}\mathrm{Re}\, V_+^n &= 2\mathrm{Re}\sum_{j<k\leq r}\left[v(z_j + z_k) + v(z_j - z_k^*)\right] \\ &\quad - \sum_{k=1}^{r} v(\sqrt{2}q_k^-) + 2r_0\mathrm{Re}\sum_{j\leq r} v(z_j + iq).\end{aligned}$$

To illustrate the form of the new Calogero-Moser potentials we note that:

$$\mathrm{Re}\,\frac{1}{\sin^2(x+iy)} = \frac{(\sin x\cosh y)^2-(\cos x\sinh y)^2}{((\sin x)^2+(\sinh y)^2)^2},$$
$$\mathrm{Re}\,\frac{1}{\sinh^2(x+iy)} = \frac{(\sinh x\cos y)^2-(\cosh x\sin y)^2}{((\sinh x)^2+(\sin y)^2)^2}.$$

Together with

$$\omega_{\mathbb{R}} = \sum_{k=1}^{r}\left(dp_k^+\wedge dq_k^+ - dp_k^-\wedge dq_k^-\right) - r_0 dp\wedge dq \qquad (64)$$

they define the real form dynamics which is obviously not confined to the standard Calogero-Moser models.

More explicitly:
for $n=2r+1$ and $\mathfrak{g}\simeq A_{n-1}$:

$$\begin{aligned} H_{\mathrm{CM},\mathbb{R}} = {}& \frac{1}{2}\sum_{k=1}^{r}\left[(p_k^+)^2-(p_k^-)^2\right] - \frac{1}{2}(p_{r+1}^-)^2 \\ &+2g^2\sum_{i<j}^{r}\mathrm{Re}\,\left[v(z_i-z_j)+v(z_i+z_j^*)\right] \\ &+g^2\sum_{j=1}^{r}v(\sqrt{2}q_j^+) + 2g^2\sum_{j=1}^{r}\mathrm{Re}\,v(z_i-iq_{r+1}^-). \end{aligned}$$

for $n=2r$ and $\mathfrak{g}\simeq A_{n-1}$:

$$\begin{aligned} H_{\mathrm{CM},\mathbb{R}} = {}& \frac{1}{2}\sum_{k=1}^{r}\left[(p_k^+)^2-(p_k^-)^2\right] + g^2\sum_{j=1}^{r}v(\sqrt{2}q_j^+) \\ &+2g^2\sum_{i<j}^{r}\mathrm{Re}\,\left[v(z_i-z_j)+v(z_i+z_j^*)\right]. \end{aligned}$$

for $n=2r+1$ and $\mathfrak{g}\simeq BC_n$:

$$\begin{aligned} H_{\mathrm{CM},\mathbb{R}} = {}& \frac{1}{2}\sum_{k=1}^{r}\left[(p_k^+)^2-(p_k^-)^2\right] - \frac{1}{2}(p_{r+1}^-)^2 \\ &+g^2\sum_{j=1}^{r}\left[v(\sqrt{2}q_j^+)-v(\sqrt{2}q_j^-)\right] \\ &+2g^2\sum_{i<j}^{r}\mathrm{Re}\,\left[v(z_i-z_j)\right. \\ &\left.+\,v(z_i+z_j^*)+v(z_i+z_j)+v(z_i-z_j^*)\right] \\ &+2g^2\sum_{j=1}^{r}\mathrm{Re}\,\left[v(z_i-iq_{r+1}^-)+v(z_i+iq_{r+1}^-)\right] \\ &-g_S^2 v(q_{r+1}^-) - g_L^2 v(2q_{r+1}^-) \\ &+2g_S^2\sum_{j=1}^{r}\mathrm{Re}\,v(z_j) + 2g_L^2\sum_{j=1}^{r}\mathrm{Re}\,v(2z_j) \end{aligned}$$

for $n=2r$ and $\mathfrak{g}\simeq BC_n$:

$$\begin{aligned} H_{\mathrm{CM},\mathbb{R}} = {}& \frac{1}{2}\sum_{k=1}^{r}\left[(p_k^+)^2-(p_k^-)^2\right] \\ &+g^2\sum_{j=1}^{r}\left[v(\sqrt{2}q_j^+)-v(\sqrt{2}q_j^-)\right] \\ &+2g^2\sum_{i<j}^{r}\mathrm{Re}\,\left[v(z_i-z_j)\right. \\ &\left.+\,v(z_i+z_j^*)+v(z_i+z_j)+v(z_i-z_j^*)\right] \\ &+2g_S^2\sum_{j=1}^{r}\mathrm{Re}\,v(z_j) + 2g_L^2\sum_{j=1}^{r}\mathrm{Re}\,v(2z_j). \end{aligned}$$

Note that: $v(iy)=-v(y)$.

Example 3. In the case of involution $\mathcal{C}_2 = S_{e_1-e_2}$:

$$\mathcal{C}_2(q_1)=q_2,\ \mathcal{C}_2(q_2)=q_1,\ \mathcal{C}_2(p_1)=p_2,\ \mathcal{C}_2(p_2)=p_1,$$

we obtain for $\mathfrak{g}\simeq BC_n$:

$$\begin{aligned} H_{\mathrm{CM},\mathbb{R}} = {}& \frac{1}{2}\left[(p_1^+)^2-(p_1^-)^2+\sum_{k=3}^{n}(p_k^+)^2\right] \\ &+g_S^2\left[2\mathrm{Re}\,v(z_1)+\sum_{j=3}^{n}\mathrm{Re}\,v(q_j)\right] \\ &+g^2\left\{2\sum_{j=3}^{n}\mathrm{Re}\,\left[v(z_1-q_j^+)+v(z_1+q_j^+)\right]\right. \\ &-v(\sqrt{2}q_1^-)+v(\sqrt{2}q_1^+) \\ &\left.+\sum_{3\le k<j}^{n}\left[v(q_k-q_j)+v(q_k+q_j)\right]\right\} \\ &+g_L^2\left[2\mathrm{Re}\,v(2z_1)+\sum_{j=3}^{n}\mathrm{Re}\,v(2q_j)\right] \end{aligned}$$

Example 4. In the case of involution $\mathcal{C}_3 = -\mathbb{1}$:

$$\mathcal{C}_2(q_k)=-q_k,\ \mathcal{C}_2(p_k)=-p_k,$$

$$q_k^-=q_k,\ p_k^-=p_k,\ \dim\mathcal{M}_-=2n.$$

We obtain for Toda chains with the algebra $A_{n-1}\simeq sl(n)$:

$$H_{\mathrm{TC}} = \frac{1}{2}\sum_{k=1}^{n}p_k^2 + \frac{m^2}{\beta^2}\sum_{k=1}^{n-1}e^{\beta(\alpha_k,\boldsymbol{q})} + c_0 e^{\beta(\alpha_0,\boldsymbol{q})},$$

where $c_0=0$ stands for conformal TC and $c_0=1$ for affine TC. Now all p_k and q_k become purely imaginary. We choose also $\beta=i\beta_0$. As a result we obtain Toda chain with purely imaginary interaction constant.

Similarly, for CMS we obtain:

A) If $v(q) = 1/q^2$ or $v(q) = 1/q^2+\omega^2 q^2$ then we obtain:

$$H_{\mathrm{CM},\mathbb{R}} = -H, \qquad \text{which is equivalent to} \qquad t \leftrightarrow -t.$$

B) If $v_2(q) = a^2/\sin^2(aq)$ and $v_3(q) = a^2/\sinh^2(aq)$ then the real form dynamics is obtained by the exchange:

$$v_2(q) \leftrightarrow -v_3(q), \qquad t \leftrightarrow -t.$$

C) If $v_4(q) = a^2\wp(aq|\omega_1,\omega_2)$ then the real form dynamics is obtained by the exchange:

$$v_4(q) \leftrightarrow -v_4'(q), \qquad t \leftrightarrow -t.$$

where $v_4'(q) = -a^2\wp(aq|i\omega_1, i\omega_2)$.

Remark 4. Several of the RHF of CMS have been obtained earlier by Calogero [18] using the so-called 'duplication procedure'. Our results above show that each 'duplication' is related to a Cartan-like automorphism of the relevant Lie algebra. This fact can be used to classify all inequivalent RHF of CMS.

Example 5. An example from general relativity: transition from $SO(3)$ to $SO(2,1)$ symmetry.

Recall that the geodesic Schwarzschild flow, corresponding to a metric $g = g_{ij}dx_i dx_j$ on $\mathcal{M}$, can be viewed as the flow, generated by the Hamiltonian field X_H with $H = \frac{1}{4}g^{ij}p_ip_j$ (p_i's are conjugated to x_i's coordinates on $T^*\mathcal{M}$).

For example, the geodesic flow for the Schwarzschild metric

$$g_1 = \epsilon_1\left(\frac{rc^2-2MG}{rc^2}dt^2 - \frac{rc^2}{rc^2-2MG}\right)dr^2 + \epsilon_2 r^2\left(d\vartheta^2 + \sin^2\vartheta\, d\varphi^2\right), \tag{65}$$

where $\epsilon_1 = \pm 1$, $\epsilon_2 = \pm 1$, is described by the (completely integrable) Hamiltonian

$$\mathcal{H}_1 = \frac{1}{4}\left[\epsilon_1\left(\frac{rc^2-2MG}{rc^2}p_t^2 - \frac{rc^2-2MG}{rc^2}p_r^2\right) + \frac{\epsilon_2}{r^2}\left(p_\vartheta^2 + \frac{1}{\sin^2\vartheta}p_\varphi^2\right)\right]. \tag{66}$$

The complexification and real projection procedure leads to the following Hamiltonian

$$\mathcal{H}_2 = \frac{1}{4}\left[\epsilon_1\left(\frac{rc^2-2MG}{rc^2}p_t^2 - \frac{rc^2-2MG}{rc^2}p_r^2\right) - \frac{\epsilon_2}{r^2}\left(p_\vartheta^2 + \frac{1}{\sinh^2\vartheta}p_\varphi^2\right)\right] \tag{67}$$

which (is still completely integrable and) is associated with the metric

$$g_2 = \epsilon_1\left(\frac{rc^2-2MG}{rc^2}dt^2 - \frac{rc^2}{rc^2-2MG}\right)dr^2 - \epsilon_2 r^2\left(d\vartheta^2 + \sinh^2\vartheta\, d\varphi^2\right), \tag{68}$$

solution of vacuum Einstein field equations [19,20].

The above notation for coordinates might be misleading because, in g_1 and $\mathcal{H}_1$, $r \in]0,\infty[$, $\vartheta \in]-\pi,\pi[$, $\varphi \in [0,2\pi[$ denote spherical coordinates, while, in g_2 and $\mathcal{H}_2$, $r \in]0,\infty[$, $\vartheta \in R$, $\varphi \in [0,2\pi[$ denote pseudo-spherical coordinates.

The 2-dimensional surfaces endowed with metric g_2 restricted by ($r, t =$ const) may be identified with one of the sheets of the two-sheeted space-like hyperboloid. They are also known as *pseudo-spheres*.

The pseudo-sphere is a surface with constant negative Gaussian curvature $\mathcal{R} = -1/r^2$. It can be globally embedded in a 3-dimensional Minkowskian space. Let y_1, y_2, y_3 denote the coordinates in the Minkowskian space, where the separation from the origin is given by $y^2 = -y_1^2 + y_2^2 + y_3^2$. These coordinates are connected to the pseudo-spherical coordinates (r,ϑ,φ) by:
$y_1 = r\cosh\vartheta, \quad y_2 = r\sinh\vartheta\cos\varphi, \quad y_3 = r\sinh\vartheta\sin\varphi.$
The equation $y^2 = -r^2$, i.e. the locus of points equidistant from the origin, specifies a hyperboloid of two sheets intersecting the y_1 axis at the points $\pm r$ called *poles* in analogy with the sphere. Either sheet (say the upper sheet) models an infinite spacelike surface without a boundary; hence, the Minkowski metric becomes positive definite (Riemannian) upon it. This surface has constant Gaussian curvature ($\mathcal{R} = -1/r^2$), and it is the only simply connected surface with this property. Other embeddings of the pseudo-sphere in the 3-dimensional Euclidean space are also available, for example it can be regarded as the 2-dimensional surface generated by the tractrix, but they are not global.

It is worth noting that metrics g_1 and g_2 are $so(3)$-invariant and $so(2,1)$-invariant[2], respectively, and that geodesic flows, corresponding to the model Ricci-flat metrics, associated with 3-dimensional Killing algebras, are integrable. To see that one may observe that, for instance, Hamiltonian $\mathcal{H}_2$ possesses five independent first integrals

$$\mathcal{H}_2, \quad p_\tau, \quad p_\vartheta^2 + \frac{1}{\sinh^2\vartheta}p_\varphi^2, \quad \mathcal{I}_1, \quad \mathcal{I}_2$$

where $\mathcal{I}_1$ and $\mathcal{I}_2$ are generators of a noncommutative two-dimensional subalgebra of $so(2,1)$, for example

$$\mathcal{I}_1 = \left[\left(1+\sqrt{2}\right)\cos\varphi + \sin\varphi\right]p_\vartheta + \left[1+\sqrt{2} + \coth\vartheta\left(\cos\varphi - \left(1+\sqrt{2}\right)\sin\varphi\right)\right]p_\varphi$$

$$\mathcal{I}_2 = \sqrt{2}\left[\cos\varphi + \sin\varphi\right]p_\vartheta + \left[2 + \sqrt{2}\coth\vartheta\left(\cos\varphi - \sin\varphi\right)\right]p_\varphi.$$

[2] In the pseudo-spherical coordinates, the $so(2,1)$ Lie algebra $[X_1,X_2] = X_3$, $[X_2,X_3] = -X_1$, $[X_3,X_1] = -X_2$,, is spanned by $X_1 = \sin\varphi\partial_\vartheta + \cos\varphi\coth\vartheta\partial_\varphi$, $X_2 = -\cos\varphi\partial_\vartheta + \sin\varphi\coth\vartheta\partial_\varphi$, $X_3 = \partial_\varphi$ [21].

The 5 first integrals span a rank 3 Lie algebra $\mathcal{A}$, and since

$$\mathrm{rank}\,\mathcal{A} + \dim\mathcal{A} = \dim T^*\mathcal{M} \quad \text{and} \quad \mathrm{rank}\,\mathcal{A} < \dim\mathcal{A}$$

the system is noncommutatively integrable in the sense [22,23].

Geodesic flows corresponding to the metrics invariant for a 3-dimensional Lie algebra are discussed in [20]. Note also that the above proposition is no more valid for the geodesic flow of Ricci-flat metrics with nonextendable two-dimensional Killing algebras.

7 Lax operators

It is well known that a number of completely integrable systems admit Lax representation. By it we mean the existence of two matrices $L(\mathbf{p},\mathbf{q})$ and $M(\mathbf{p},\mathbf{q})$ depending explicitly on the dynamical variables and such that the Lax equation

$$\frac{dL}{dt} = [L, M] \tag{69}$$

is equivalent to the corresponding equations of motion of $\mathcal{H}$. Let us also remind that from (69) there follows immediately that the eigenvalues ζ_s of L are integrals of the motion in involution. One can use also as such

$$I_k = \mathrm{tr}\, L^k(\mathbf{p},\mathbf{q}) = \sum_{s=1}^{n} \zeta_s^k, \qquad k = 1,\ldots,n, \tag{70}$$

and as a rule the Hamiltonian H is a linear combination of I_k's; for example, for the Toda chain and for the CMS $H = 2\,I_2$.

If the matrices L and M are analytic (meromorphic) functions of p_k and q_k then obviously these properties will hold true also for the integrals I_k. The analytic properties of ζ_s as functions of p_k and q_k require additional considerations. Indeed, ζ_s are roots of the corresponding characteristic equation which is of order n so the mapping from p_k and q_k to ζ_s may have essential singularities.

The Lax representation ensures also the separability of all I_k (including H) in terms of ζ_s. For some of the best known cases like for the Toda chain, the matrices L and M take values in the normal real form of some simple Lie algebra $\mathfrak{g}$. This means that L is a linear combination of the Cartan-Weyl generators with real valued coefficients.

The complexification procedure renders p_k and q_k complex and as a result both L and M become complex-valued:

$$L \to L^{\mathbb{C}} = L_0 + iL_1,\ M \to M^{\mathbb{C}} = M_0 + iM_1, \quad L, M \in \mathfrak{g}^{\mathbb{C}}.$$

Note however, that the complexified Lax equation

$$\frac{dL^{\mathbb{C}}}{dt} = [L^{\mathbb{C}}, M^{\mathbb{C}}], \tag{71}$$

can again be written down in terms of purely real Lax matrices with doubled dimension:

$$\frac{d\mathbf{L}}{dt} = [\mathbf{L}, \mathbf{M}], \quad \mathbf{L} = \begin{pmatrix} L_0 & L_1 \\ -L_1 & L_0 \end{pmatrix}, \quad \mathbf{M} = \begin{pmatrix} M_0 & M_1 \\ -M_1 & M_0 \end{pmatrix}. \tag{72}$$

This fact we will use below when we discuss generalizations of our approach.

Let us now consider the Lax representation for the Toda chain related to the simple Lie algebra $\mathfrak{g}$ with rank n:

$$\begin{aligned} L_{\mathrm{TC}} &= \sum_{k=1}^{n} p_k H_k \\ &\quad + \sum_{k=1}^{n} a_k (E_{\alpha_k} + E_{-\alpha_k}) + c_0 a_0 (E_{\alpha_0} + E_{-\alpha_0}), \\ M_{\mathrm{TC}} &= \sum_{k=1}^{n} a_k (E_{\alpha_k} - E_{-\alpha_k}) + c_0 a_0 (E_{\alpha_0} - E_{-\alpha_0}), \end{aligned} \tag{73}$$

where α_k is the set of simple roots of $\mathfrak{g}$, α_0 is the minimal root of $\mathfrak{g}$,

$$a_k = \frac{1}{2}\exp((\boldsymbol{q}, \alpha_k)), \tag{74}$$

and c_0 was introduced in (55) above. By E_α and H_k above we denote the Cartan-Weyl generators of $\mathfrak{g}$; they satisfy the commutation relations:

$$\begin{aligned} [H_k, E_\alpha] &= (\alpha, e_k) E_\alpha, \\ [E_\alpha, E_{-\alpha}] &= H_\alpha, \\ [E_\alpha, E_\beta] &= N_{\alpha,\beta} E_{\alpha+\beta}, \end{aligned} \tag{75}$$

where α and $\beta \in \Delta$ are any two roots of $\mathfrak{g}$ and $N_{\alpha,\beta} = 0$ if $\alpha + \beta \notin \Delta$.

The involution $\mathcal{C}$ induces an involutive automorphism $\mathcal{C}$ in the algebra $\mathfrak{g}$. Indeed, both $\boldsymbol{p}$ and $\boldsymbol{q}$ can be viewed as vectors in the root space $\mathbb{E}^n \simeq \mathfrak{h}^*$ which is dual to the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$. Besides, $\mathcal{C}$ must preserve the Toda chain Hamiltonian:

$$H_{\mathrm{TC}} = \frac{1}{2}\mathrm{tr}\, L_{\mathrm{TC}}^2 = \frac{1}{2}\sum_{k=1}^{n} p_k^2 + \sum_{k=1}^{n} a_k^2 + c_0^2 a_0^2. \tag{76}$$

This means that $\mathcal{C}$ in the conformal (resp. affine) case must preserve the system of simple (resp. admissible) roots of $\mathfrak{g}$. This allows one to associate with the involutive automorphism $\mathcal{C}$ an involutive automorphism $\mathcal{C}^\#$ of the algebra $\mathfrak{g}$. Obviously, if $\mathcal{C}$ is defined on $\mathcal{M}$ by

$$\mathcal{C}(p_k) = \sum_{s=1}^{n} c_{ks} p_s, \qquad \mathcal{C}(q_k) = \sum_{s=1}^{n} c_{ks} q_s, \tag{77}$$

where c_{ks} are such that $\sum_{s=1}^{n} c_{ks} c_{sm} = \delta_{km}$ then the action of $\mathcal{C}^\#$ on $\mathfrak{h}$ can be determined by duality as:

$$\mathcal{C}^\#(H_k) \equiv H_{\mathcal{C}(e_k)} = \sum_{s=1}^{n} c_{ks} H_s. \tag{78}$$

On the root vectors E_α the automorphism $\mathcal{C}^\#$ acts as follows:

$$\mathcal{C}^\#(E_\alpha) = n_\alpha E_{\mathcal{C}^\#(E_\alpha)}, \qquad n_\alpha = \pm 1. \tag{79}$$

An well known fact [9] is that the subgroup of the automorphism group preserving $\mathfrak{h}$ is determined by $\mathrm{Ad}_{\,\mathfrak{h}} \otimes$

$W_{\mathfrak{g}} \otimes V_{\mathfrak{g}}$, where Ad $_{\mathfrak{h}}$ is the subgroup of inner automorphisms by elements of the Cartan subgroup, $W_{\mathfrak{g}}$ is the Weyl group and $V_{\mathfrak{g}}$ is the group of outer automorphisms. $V_{\mathfrak{g}}$ is isomorphic to the symmetry group of the Dynkin diagram of $\mathfrak{g}$.

Since the involution $\mathcal{C}$ for the conformal Toda chain must preserve the set of simple roots of $\mathfrak{g}$ it must be related to a symmetry of the Dynkin diagram; therefore $\mathcal{C}^{\#}$ must correspond to an outer automorphism of $\mathfrak{g}$. For the affine Toda chains $\mathcal{C}^{\#}$ must be related to a symmetry of the extended Dynkin diagram which allows for a larger set of choices for $\mathcal{C}^{\#}$. These symmetries have been classified in [24].

Remark 5. The involution $\mathcal{C}^{\#}$ dual to $\mathcal{C}$ in equation (56) is the outer automorphism of $sl(n)$.

Analogously we consider the Lax representation (69) of the Calogero-Moser models is of the form [17]:

$$L(\boldsymbol{p},\boldsymbol{q}) = \sum_{j=1}^{n} p_j H_j + i \sum_{\alpha\in\Delta} g_\alpha x((\boldsymbol{q},\alpha))E_\alpha, \qquad (80)$$

$$M(\boldsymbol{p},\boldsymbol{q}) = \sum_{j=1}^{n} z_j H_j + \sum_{\alpha\in\Delta} g_\alpha y((\boldsymbol{q},\alpha))E_\alpha, \qquad (81)$$

The constant g_α depends only on the length of the root α. The three functions $x(q)$, $y(q)$ and $z(q)$ satisfy a set of functional equations

$$y(q) = -x'(q), \qquad z(\xi) = \frac{x''(\xi)}{2x(\xi)}, \qquad (82)$$

$$x(\xi)x'(\eta) - x(\eta_x'(\xi) = x(\xi+\eta)[z(\xi) - z(\eta)], \qquad (83)$$

The Hamiltonian $H = \frac{1}{2}\mathrm{tr}\, L^2$ and the function $v(q)$ are related to them by:

$$v(q) = -x(q)x(-q).$$

The solutions to these equations are given by [16–18]:

$$x(\xi) = \begin{cases} \dfrac{1}{\xi}, & \text{I} \\ a\,\coth(a\xi),\quad a\,\sinh^{-1}(a\xi), & \text{II} \\ a\,\mathrm{cotan}\,(a\xi),\ a\,\sin^{-1}(a\xi), & \text{III} \\ \dfrac{a\,\mathrm{cn}\,(a\xi)}{\mathrm{sn}\,(a\xi)},\quad \dfrac{a\,\mathrm{dn}\,(a\,\xi)}{\mathrm{sn}\,(a\xi)},\quad \dfrac{a}{\mathrm{sn}\,(a\xi)} & \text{IV} \end{cases} \qquad (84)$$

then the corresponding $v(\xi)$ provide the choices in (63).

The involution $\mathcal{C}^{\#}$ induces a Cartan involution $\underset{\sim}{C}$ on $\mathfrak{g}^{\mathbb{C}}$ by $\underset{\sim}{C}(Z) = -C(Z^\dagger)$ has all the properties of Cartan involution. As a result the invariance condition with respect to $\underset{\sim}{C}$ restricts to a real form of $\mathfrak{g}$. The invariance condition for $\widetilde{L}$ has the form:

$$\underset{\sim}{C}\left(L(C^{\#}(\boldsymbol{p}), C^{\#}(\boldsymbol{q}), g\right) = L(\boldsymbol{p},\boldsymbol{q},-g). \qquad (85)$$

For the case of the example above with $\mathfrak{g} \simeq sl(n)$ and $x(q) = -x(-q)$ this can be easily shown by realizing that $C(X)$ can be written in the form $C(X) = TXT^{-1}$ where $T = \sum_{k=1}^{n} E_{k,\bar{k}}$. One can check that (85) and an analogous relation on $M(\boldsymbol{p},\boldsymbol{q},g)$ leave the Lax representation invariant. Compare this with Mikhailov reduction group [25]. In this example we miss the spectral parameter, but we also have the interaction constant g and (85) involves nontrivial action also on g.

The substantial difference as compare to the Toda chain case is that the Calogero-Moser Hamiltonian is invariant with respect to the Weyl group of $\mathfrak{g}$ as well as with respect to the group of outer automorphisms of $\mathfrak{g}$. Therefore any involutive element of $W_{\mathfrak{g}} \otimes V_{\mathfrak{g}}$ can be used to construct a RHF of the CMS. Obviously two such RHF's will be equivalent if the corresponding automorphisms $\mathcal{C}_1$ and $\mathcal{C}_2$ belong to the same conjugacy class of $W_{\mathfrak{g}}$.

This means that the number of inequivalent choices for the involution $\mathcal{C}$ and therefore, the number of inequivalent RHF of the Calogero-Moser systems is much bigger. To classify all of them one has to consider the equivalence classes of $W(\mathfrak{g})$ and pick up just one element from each class of second order elements. It would be also natural to relate to each Satake diagram of $\mathfrak{g}$, or a subalgebra of $\mathfrak{g}$ a RHF of CMS. A detailed study will be reported elsewhere.

The Cartan subalgebra $\mathfrak{h}$ and the algebra $\mathfrak{g}$, just like $\mathcal{M}$, can be splitted into direct sums:

$$\mathfrak{h} = \mathfrak{h}_+ \oplus \mathfrak{h}_-, \qquad \mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-, \qquad (86)$$

compatible with the involution $\mathcal{C}^{\#}$; i.e.:

$$\mathcal{C}^{\#}(X) = X, \quad \text{for any}\ \ X \in \mathfrak{h}_+,\ \ \text{resp., for any}\ \ X \in \mathfrak{g}_+,$$
$$\mathcal{C}^{\#}(Y) = -Y, \quad \text{for any}\ \ Y \in \mathfrak{g}_-,\ \ \text{resp., for any}\ \ X \in \mathfrak{h}_+.$$

If we compose $\mathcal{C}^{\#}$ with $-^\dagger$ we obtain the Cartan involution $\widetilde{\mathcal{C}^{\#}}$ which selects a real form of the algebra $\mathfrak{g}_{\mathbb{R}}$.

8 Dynamics of the Toda chain and its RHF

In this Section on the example of the Toda chains, we briefly discuss how the two basic steps in constructing the RHF's may affect the dynamical regimes of the Hamiltonian systems. These two steps are:

a) complexification and,

b) imposing the involution $\tilde{\mathcal{C}}$.

The dynamical regimes of the CTC were studied recently from the point of view of their applications to the adiabatic N-soliton interactions [1]. It was already known that CTC has a much richer class of dynamical regimes than the real TC. This shows that step a) (the complexification) changes drastically the character of the dynamics of out initial Hamiltonian system.

Indeed, it is well known [26,27] that the solutions of the conformal Toda chains are parametrized by the eigenvalues ζ_j and the first components r_j of the normalized eigenvectors of the Lax matrix:

$$L\boldsymbol{w}_j = \zeta_j \boldsymbol{w}_j, \qquad r_j = \boldsymbol{w}_j^{(1)}, \qquad (87)$$

$$(\boldsymbol{w}_j, \boldsymbol{w}_k) \equiv \sum_{s=1}^{n} \boldsymbol{w}_j^{(s)} \boldsymbol{w}_k^{(s)} = \delta_{jk}. \qquad (88)$$

For the Toda chain related to the algebra $sl(n)$ these solutions take the form:

$$q_1(t) = \ln A_1(t), \qquad A_1(t) = \sum_{j=1}^{n} r_j^2 e^{-2\zeta_j t},$$

$$q_k(t) = \ln \frac{A_{k+1}}{A_k}, \qquad k = 2, 3, \dots, n, \tag{89}$$

$$A_k(t) = \sum_{j_1 < \dots < j_k} W^2(j_1, \dots, j_k) r_{j_1}^2 \dots r_{j_k}^2 \times \exp(-2(\zeta_{j_1} + \dots + \zeta_{j_k})t) \tag{90}$$

$$A_n = \prod_{k=1}^{n} r_j^2 W^2(1, 2, \dots, n), \qquad A_{n+1} = 1,$$

where $W_{j_1,\dots,j_k}$ is the Wandermonde determinant:

$$W_{j_1,\dots,j_k} = \det \begin{pmatrix} 1 & 1 & \dots & 1 \\ \zeta_{j_1} & \zeta_{j_2} & \dots & \zeta_{j_k} \\ \vdots & \vdots & \ddots & \vdots \\ \zeta_{j_1}^{k-1} & \zeta_{j_2}^{k-1} & \dots & \zeta_{j_k}^{k-1} \end{pmatrix}. \tag{91}$$

It is well known that for the real Toda chain:
1) all ζ_j are real and pair-wise different;
2) all r_j are also real [28].

This ensures that $A_k(t)$ are positive for all t and therefore the solutions $q_k(t)$ are regular for all t. In addition one can evaluate the asymptotic behavior of $q_k(t)$ for large $|t|$. If we assume that the eigenvalues ζ_k satisfy the sorting condition:

$$\zeta_1 > \zeta_2 > \dots > \zeta_n, \tag{92}$$

then we get the result:

$$\lim_{t \to \pm\infty} (q_k(t) - v_k^{\pm} t) = \beta_k^{\pm}, \tag{93}$$

$$v_k^- = -2\zeta_k, \qquad v_k^+ = -2\zeta_{n+1-k}; \tag{94}$$

for the constants $\beta_k^{\pm}$ one can derive explicit expressions in terms of ζ_j.

If we interpret $q_k(t)$ as the trajectory of the kth particle then $2\zeta_j$ characterize their asymptotic velocity. The property 1) above means that the (real) Toda chain allows only for (non-compact) asymptotically free motion of the particles.

The set of coefficients $\{\zeta_j, \ln r_j\}$ are the action-angle variables in the generalized sense, introduced in the beginning of Section 5. Of course they are convenient for solving the TC model because: i) they satisfy canonical Poisson brackets and ii) the TC Hamiltonian takes the simple form:

$$H_{\mathrm{TC}} = \sum_{j=1}^{n} 2\zeta_j^2. \tag{95}$$

Therefore the equations of motion for $\{\zeta_j, \ln r_j^2\}$ take the form:

$$\frac{d\zeta_j}{dt} = 0, \qquad \frac{d \ln r_j^2}{dt} = -2\zeta_j, \qquad j = 1, \dots, n. \tag{96}$$

The situation changes substantially after the complexification, see [2]. In fact the same formulae (89), (90) provide the solution also for the complex case. But now we have neither of the properties 1) or 2). The eigenvalues ζ_j, as well as r_j become complex:

$$\zeta_j = \kappa_j + i\eta_j, \qquad \rho_j + i\phi_j = \ln r_j e^{q_1(0)}, \qquad j = 1, \dots, n, \tag{97}$$

where

$$e^{-nq_1(0)} = A_n = \prod_{j=1}^{n} r_j^2 W^2(1, \dots, n).$$

This substantially modifies the properties of the solutions.

For CTC one can derive the analogs of the relations (93):

$$\lim_{t \to \pm\infty} (q_k(t) - v_k^{\pm} t) = \beta_k^{\pm}, \tag{98}$$

$$v_k^- = -2\kappa_k, \qquad v_k^+ = -2\kappa_{n+1-k}; \tag{99}$$

which shows that now it is the set of real parts $\{\kappa_j\}$ of the eigenvalues ζ_j which determine the asymptotic velocities. Even if we assume[3] that $\zeta_j \neq \zeta_k$ for $j \neq k$ then we still may have $\kappa_j = \kappa_k$; note that equal asymptotic velocities allow for the possibility of bound states. Skipping the details (see [1,2]) we just list the different types of asymptotic regimes for the CTC which are determined by the so-called sorting condition:

$$\kappa_1 \geq \kappa_2 \geq \dots \geq \kappa_n. \tag{100}$$

This sorting condition differs from (92) for the real TC in that it allows equalities. Therefore the CTC allows several non-degenerate regimes:

A) Asymptotically free regime: $\kappa_j \neq \kappa_k$ for $j \neq k$;

B) n-particle bound state for $\kappa_1 = \kappa_2 = \dots = \kappa_n$;

C) A number of mixed regimes when we have two or more groups of equal κ_j's;

Another difference with respect to the real TC case is that $A_k(t)$ are no more positive definite. For some choices of the ζ_j and r_j they may vanish even for finite values of $t = t_0$ which means that $q_k(t)$ may develop singularity for $t \to t_0$.

Let us now analyze the effect of the involutions $\mathcal{C}$ and $\tilde{\mathcal{C}} \equiv \mathcal{C} \circ *$ on the spectral data $\{\zeta_k, \ln r_k\}$ of L. It will allow us to describe the dynamical regimes of the RHF of CTC. Equation (56) leads to:

$$\tilde{\mathcal{C}}(q_k) = -q_{n+1-k}^*, \qquad \tilde{\mathcal{C}}(p_k) = -p_{n+1-k}^*, \tag{101}$$

and consequently:

$$\tilde{\mathcal{C}}(b_k) = -b_{n+1-k}^*, \qquad \tilde{\mathcal{C}}(a_k) = a_{n-k}^*. \tag{102}$$

[3] In principle we may have $\zeta_j = \zeta_k$ for $k \neq j$ which leads to degenerate solutions; we will not consider such cases below, see e.g. [2].

These constraints mean that the Lax matrix L satisfies:

$$S_0 L(\tilde{\mathcal{C}}(p_k), \tilde{\mathcal{C}}(q_k)) S_0^{-1} = -L(p_k, q_k)^\dagger, \qquad (103)$$

$$S_0 = \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & -1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & (-1)^{n-1} & \dots & 0 & 0 \\ (-1)^n & 0 & \dots & 0 & 0 \end{pmatrix}. \qquad (104)$$

This means that the eigenvalues and the matrix of eigenvectors of L satisfy:

$$S_0 Z S_0^{-1} = -Z^*, \qquad Z = \operatorname{diag}(\zeta_1, \dots, \zeta_n) \quad \text{or} \quad \tilde{\mathcal{C}}(\zeta_k) = -\zeta_{n+1-k}^*. \qquad (105)$$

Technically it is a bit more difficult to derive the action of $\tilde{\mathcal{C}}(r_k)$. Indeed, this must ensure that the functions A_k satisfy

$$\tilde{\mathcal{C}}(A_k) = A_{n-k}^*, \qquad (106)$$

for all values of t. After some calculations we get that the relations

$$\rho_{n+1-k} + i\phi_{n+1-k} = -\rho_k + i\phi_k + \ln w_k, \qquad (107)$$
$$w_k = \frac{W(1, \dots, k-1, k+1, \dots, n)}{W(1, 2, \dots, n)}$$

ensure that $B_{n-k}^* = B_k$ where

$$B_k(t) = A_k(t) e^{-k q_1(0)}. \qquad (108)$$

Let us illustrate this by first putting $n = 4$ and choosing

$$\begin{aligned} &\zeta_1 = -\zeta_4^* = \kappa_1 + i\eta_1, \qquad \zeta_2 = -\zeta_3^* = \kappa_1 - i\eta_1, \\ &\rho_1 = \rho_0 + i\phi_0, \qquad \rho_3 = -\rho_0 - i(\phi_0 + \alpha + \frac{\pi}{2}), \qquad (109) \\ &\rho_2 = \rho_0 - i\phi_0, \qquad \rho_4 = -\rho_0 + i(\phi_0 + \alpha + \frac{\pi}{2}). \end{aligned}$$

With these notations we can write down the solution for the $sl(4)$-CTC in the form:

$$q_1(t) = \ln B_1(t), \quad q_2(t) = \ln \frac{B_2(t)}{B_1(t)}, \qquad (110)$$
$$q_3(t) = \ln \frac{B_3(t)}{B_2(t)} = -q_2^*(t), \qquad q_4(t) = -\ln B_3(t) = -q_1^*(t),$$

where

$$\begin{aligned} B_1(t) &= \frac{1}{2^5 \kappa_1 \eta_1 \sqrt{\kappa_1^2 + \eta_1^2}} \left[e^{-2\kappa_1 t + 2\rho_0} \cos(2\eta_1 t - 2\phi_0) \right. \\ &\quad \left. - e^{2\kappa_1 t - 2\rho_0} \cos(2\eta_1 t - 2\phi_0 - 2\alpha) \right] \qquad (111) \\ B_2(t) &= \frac{-1}{2^7 \kappa_1^2 \eta_1^2 (\kappa_1^2 + \eta_1^2)} \left[\eta_1^2 \cosh(4\eta_1 t - 4\phi_0) \right. \\ &\quad \left. + \kappa_1^2 \cos(4\eta_1 t - 4\phi_0 - 2\alpha) + \kappa_1^2 + \eta_1^2 \right], \\ B_3(t) &= B_1^*, \qquad \alpha_1 = \arg \zeta_1 = \arctan \frac{\eta_1}{\kappa_1}. \end{aligned}$$

The choice of $\zeta_1 = \zeta_2^*$ in (109) combined with the involution $\zeta_1 = -\zeta_4^*$, $\zeta_2 = -\zeta_3^*$ ensures that $q_1(t)$ and $q_2(t)$ will have equal asymptotic velocities and will form a bound state. Indeed, from (110) and (111) we get:

$$\begin{aligned} \lim_{t \to \pm\infty} \left(\frac{1}{2}(q_1(t) + q_2(t)) \mp 2\kappa_1 t \right) &= \frac{i\pi}{2} \mp 2\rho_0 \\ &- \frac{1}{2} \ln 2^8 \kappa_1^2 (\kappa_1^2 + \eta_1^2) + \mathcal{O}\left(e^{-4\kappa_1 |t|} \right), \qquad (112) \end{aligned}$$

which means that the center of mass of the particles $q_1(t)$ and $q_2(t)$ for $t \to \pm\infty$ undergoes non-compact asymptotically free motion with asymptotic velocities $\pm 2\kappa_1$. At the same time the relative asymptotic motion is compact:

$$\begin{aligned} q_1(t) - q_2(t) &\underset{t \to \pm\infty}{\longrightarrow} i\pi - \ln 4\eta_1^2 + \ln \cos^2(\Phi^\pm(t)) \qquad (113) \\ &+ \mathcal{O}\left(e^{-2\kappa_1 |t|} \right), \\ \Phi^+(t) &= 2\eta_1 t - 2\phi_0 - 2\alpha_1, \qquad \Phi^-(t) = 2\eta_1 t - 2\phi_0. \end{aligned}$$

The energy of these RHF of $sl(4)$-CTC is given by:

$$H_{sl(4)} = 8(\kappa_1^2 - \eta_1^2). \qquad (114)$$

Due to the symmetry the 'particles' $q_3(t)$ and $q_4(t)$ also form a bound state; the corresponding formulae easily follow from (112) and (113) and $q_4(t) = -q_1^*(t)$ and $q_3(t) = -q_2^*(t)$.

This example demonstrates several peculiarities of the RHF dynamics. The first one is that configurations in which all n particles form a bound state is impossible. Indeed, such bound state may take place if $\kappa_j = 0$ for all $j = 1, \dots, n$. But it is easy to check that the sorting condition (100) and the symmetry of the eigenvalues (109) lead to degeneracy of the spectrum of L for $\kappa_j = 0$.

The RHF dynamics allows either asymptotically free regimes or mixed regimes with special symmetry, namely we can have only even number of bound states which move with opposite asymptotic velocities.

The above considerations on the properties of the dynamical regimes are not specific for the Toda chain. Similar conclusions can be derived also for the RHF of the Calogero-Moser systems. Here we have richer variety of RHF's due to the fact that the CMS Hamiltonian is invariant with respect to the whole Weyl group of $\mathfrak{g}$. More detailed investigation of the CMS dynamical regimes will be done elsewhere.

9 Discussion

1. The method goes also for non-integrable models, though we treated mainly integrable ones.
   Other related aspects of our method concern more complicated objects such as dynamical classical r-matrices or quantum R-matrices due to their algebraic structures will also satisfy relations of the form (85).
2. Our list of examples can be substantially extended with Ruijsenaars-Sneider models and their generalizations being among the obvious candidates.

3. The present approach can naturally be generalized also for infinite-dimensional integrable models such as the $1+1$-dimensional Toda field theories. This will allow one to construct new classes of real Hamiltonian forms of Toda field theories extending the results of [6].
4. The method can be used repeatedly: Since the complexified model can be viewed as real Hamiltonian system with $2n$ degrees of freedom, then we can start and complexify it getting a Hamiltonian system with $4n$ degrees of freedom producing a real form dynamics with $2n$ degrees of freedom.
   If the initial system allows Lax representation that depends analytically on the dynamical parameters:
   $$\frac{dL_0}{dt} = [L_0, M_0]. \tag{115}$$
   After the complexification we get complex-valued $L = L_0 + iL_1$ and $M = M_0 + iM_1$ which provide the Lax representation for the complexified system. But the complexified system can be viewed as real Hamiltonian system with $2n$ degrees of freedom. It allows Lax representation with real-valued L and M of the form (72).
5. Solutions of the initial system that depend analytically on the initial parameters 'survive' the complexification procedure. As we see from the CTC case discussed above, solutions that have been regular for all time after such procedure may develop singularities for finite t. Another substantial difference is in the asymptotic dynamical regimes.
   Projecting onto the RHF some of these singularities remain. Their asymptotic dynamical regimes need additional studies.

One of us (VSG) acknowledges financial support in part from Gruppo collegato di INFN at Salerno, Italy and project PRIN 2000 (contract 323/2002). The work of GM and GV was partially supported by PRIN SINTESI. One of us (VSG) thanks Professor F. Calogero for the hospitality at Rome University and the nice discussions concerning this paper. One of us (AK) is grateful to the Department of Physics of the University of Salerno and to INFN group at Salerno for their warm hospitality.

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